1.1: First-Order Differential Equations

Definition 1. An equation which relates an unknown function (fcn) and one of more of its derivatives is called a **differential equation** (diff eqn).

Example 1.

$$\frac{dx}{dt} = x^2 + t^2 \quad \text{or} \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

The first equation relates the time rate of change $x'(t) = dx/dt$ to the original function $x = x(t)$ and the independent (time) variable t. The second equation simply relates a function $y = y(x)$ to its first and second derivative.

Three Principle Goals of Studying Differential Equations

- To discover the differential equation that describes a specific physical situation.
- To find-either exactly of approximately- the appropriate solution to that equation.
- To interpret the solution that is found.

Example 2. Let C be a constant; i.e. $C \in \mathbb{R}$. The function

$$y = Ce^{x^2} \tag{1}$$

satisfies the differential equation

$$\frac{dy}{dx} = 2xy. \tag{2}$$

Remark. Notice that since $C \in \mathbb{R}$ is arbitrary in (1) the differential equation in (2) has infinitely many solutions.

Exercise 1. Verify that the function given in (1) satisfies the differential equation (2).

$$y = Ce^{x^2}$$

$$\frac{dy}{dx} = Ce^{x^2} \cdot 2x = y \cdot 2x = 2xy.$$

Example 3. (Newton's Law of Cooling)
If $T = T(t)$ is the temperature of a body in a medium with temperature A, then

$$\frac{dT}{dt} = -k(T - A) \tag{3}$$

for some positive constant k. In words, (3) says that the time rate of change of the temperature is proportional to the difference between the temperature of the body and the surrounding medium.

Question 1. What intuitively obvious physical property can be deduced from (3)?

$T > A$ then $\frac{dT}{dt} < 0$ $T < A$ then $\frac{dT}{dt} > 0$ When $T-A$ is large $\frac{dT}{dt}$ is large.

Example 4. (Torricelli's Law)
Torricelli's law implies that the time rate of change of volume V or water in a draining tank is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = -k\sqrt{V}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$ and $dV/dt = A \cdot (dy/dt)$. In this case, (4) becomes

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/\sqrt{A}$.

Example 5. Suppose the time rate of change of a population $P(t)$, with constant birth and death rates, is proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. We can easily verify that

$$P(t) = Ce^{kt} \tag{7}$$

is a solution to (6) for any constant C. Thus, even if the value of k is known, the differential equation (6) has infinitely many solutions.

Exercise 2. Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, in hours, that the population at time $t = 0$ was 1000, and that the population doubled after 1 hour. Use this additional information to solve for the two constants C and k.

$$C = C \cdot e^0 = P(0) = 1000 \Rightarrow C = 1000$$

$$1000e^k = P(1) = 2000 \Rightarrow P(t) = 1000\, e^{\ln(2) \cdot t}$$

$$2 = e^k \Rightarrow k = \ln 2$$

Definition 2. The condition $P(0) = 1000$ in Exercise 2 is called an **initial condition** because we will often write differential equations for which $t = 0$ is the "starting time." You will notice that the initial condition allowed us to take the infinitely many solutions from (7) and solve for the constant C (giving us a unique solution).

Exercise 3. Let C be a constant and $y = 1/(C - x)$. Verify that y satisfies the differential equation

$$\frac{dy}{dx} = y^2.$$

$$\frac{dy}{dx} = \frac{-1}{(C-x)^2} \cdot (-1) = \frac{1}{(C-x)^2} = y^2.$$

Exercise 4. Verify that the function $y(x) = 2x^{1/2} - x^{1/2} \ln x$ satisfies the differential equation

$$4x^2 y'' + y = 0 \quad \text{for all } x > 0.$$

$$y' = x^{-1/2} - \left(\frac{\ln x}{2x^{1/2}} + \frac{1}{x^{1/2}} \right) = \frac{-\ln x}{2x^{1/2}}$$

$$y'' = -\left(\frac{2x^{1/2} \cdot \frac{1}{x} - \ln x \left(x^{-1/2} \right)}{4x} \right) = -\left(\frac{2x^{-1/2} - x^{-1/2} \ln x}{4x} \right) = -\left(\frac{2x^{1/2} - x^{1/2} \ln x}{4x^2} \right) = \frac{-y}{4x^2}.$$

Thus $4x^2 y'' + y = 4x^2 \cdot \left(\frac{-y}{4x^2} \right) + y = 0.$

Exercise 5 is important.

Exercise 5. Let A and B be constants and $y = A\cos 3x + B\sin 3x$. Verify that y is a solution to the differential equation

$$y'' + 9y = 0.$$

$$y' = -3A\sin 3x + 3B\cos 3x$$
$$y'' = -9A\cos 3x - 9B\sin 3x \quad \Rightarrow \quad y'' + 9y = 0.$$

Definition 3.

(a) The **order** of a differential equation is the order of the highest derivative that appears in it. For example, exercise 3 involves a 1st order differential equation and exercise 4 involves a 2nd order differential equation.

(b) As we have seen, often there are infinitely many solutions to a differential equation if we do not specify initial conditions. For instance, consider the constants C and A, B in exercise 3 and 5, respectively. In these cases we often refer to the constants and **parameters** and the functions y (in each) as a **one-parameter family** or **two-parameter family** respectively.

(c) A differential equation is called **ordinary** if the function in question depends only on a single variable.

(d) If the function in question depends on two or more variables then the differential equation is called **partial**.

Exercise 6. Solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Refer to Exercise 3. From Exercise 3, $\frac{dy}{dx} = y^2$ has a solution $y = \frac{1}{C-x}$.

$$\frac{1}{C-1} = y(1) = 2 \Rightarrow C = \frac{3}{2} \text{ and } y = \frac{1}{\frac{3}{2}-x}.$$

Homework. 1-9, 13-21, 27-41 (odd)